Lower central series for surface braid groups

Paolo Bellingeri
Laboratoire de Mathématiques Jean Leray, UMR CNRS 6629,
Université de Nantes,
2, Rue de la Houssinière, 44072 Nantes, France.
bellingeri-p-at-univ-nantes.fr

Sylvain Gervais
Laboratoire de Mathématiques Jean Leray, UMR CNRS 6629,
Université de Nantes,
2, Rue de la Houssinière, 44072 Nantes, France.
gervais-s-at-univ-nantes.fr

John Guaschi
Laboratoire de Mathématiques Emile Picard, UMR CNRS 5580,
UFR-MIG, Université Toulouse III,
31062 Toulouse Cedex 9, France.
guaschi-at-picard.ups-tlse.fr

Abstract

We determine the lower central series and corresponding residual properties for braid groups and pure braid groups of orientable surfaces.

1 Introduction

Surface braid groups are a natural generalisation of the classical braid groups (corresponding to the case where Σ is a disc) and of fundamental groups of surfaces (corresponding to the case $n = 1$). They were first defined by Zariski during the

2000 AMS Mathematics Subject Classification: 20F14, 20F36, 57M.

1930's (braid groups on the sphere had been considered earlier by Hurwitz), were re-discovered by Fox during the 1960's, and were used subsequently in the study of mapping class groups.

We recall two definitions of surface braid groups. In Section 5.2, we shall give a third equivalent definition using mapping class groups.

Surface braid groups via configuration space. Let Σ be a connected, orientable surface. Let $F_n(\Sigma) = \Sigma^n \setminus \Delta$, where Δ is the fat diagonal, *i.e.* the set of n-tuples $x = (x_1, \ldots, x_n)$ for which $x_i = x_j$ for some $i \neq j$. The fundamental group $\pi_1(F_n(\Sigma))$ is called the *pure braid group* on n strands of the surface Σ; it shall be denoted by $P_n(\Sigma)$. There is a natural action of the symmetric group S_n on $F_n(\Sigma)$ by permutation of coordinates. We denote by $\widehat{F_n(\Sigma)}$ the quotient space $F_n(\Sigma)/S_n$. The fundamental group $\pi_1(\widehat{F_n(\Sigma)})$ is called the *braid group* on n strands of the surface Σ; it shall be denoted by $B_n(\Sigma)$.

Surface braid groups as equivalence classes of geometric braids. Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of n distinct points (*punctures*) in the interior of Σ. A *geometric braid* on Σ based at $\mathcal{P}$ is a collection $(\psi_1, \ldots, \psi_n)$ of n disjoint paths (called *strands*) on $\Sigma \times [0,1]$ which run monotonically with $t \in [0,1]$ and such that $\psi_i(0) = (p_i, 0)$ and $\psi_i(1) \in \mathcal{P} \times \{1\}$. Two braids are considered to be equivalent if they are isotopic. The usual product of paths defines a group structure on the equivalence classes of braids. This group, which is isomorphic to $B_n(\Sigma)$, does not depend on the choice of $\mathcal{P}$. A braid is said to be *pure* if $\psi_i(1) = (p_i, 1)$ for all $i = 1, \ldots, n$. The set of pure braids form a group isomorphic to $P_n(\Sigma)$.

Given a group G, we define the *lower central series* of G inductively as follows: set $\Gamma_1(G) = G$, and for $i \geq 2$, define $\Gamma_i(G) = [G, \Gamma_{i-1}(G)]$. The group G is said to be *perfect* if $G = \Gamma_2(G)$. From the lower central series of G one can define another filtration $D_1(G) \supseteq D_2(G) \supseteq \cdots$ by setting $D_1(G) = G$, and for $i \geq 2$, defining $D_i(G) = \{\, x \in G \mid x^n \in \Gamma_i(G) \text{ for some } n \in \mathbb{N}^* \,\}$. Following Garoufalidis and Levine [GLe], this filtration is called the *rational lower central series* of G.
Following P. Hall, for any group-theoretic property $\mathcal{P}$, a group G is said to be *residually* $\mathcal{P}$ if for any (non-trivial) element $x \in G$, there exists a group H with the property $\mathcal{P}$ and a surjective homomorphism $\varphi\colon G \longrightarrow H$ such that $\varphi(x) \neq 1$. It is well known that a group G is residually nilpotent if and only if $\bigcap_{i\geq 1} \Gamma_i(G) = \{1\}$. On the other hand, a group G is residually torsion-free nilpotent if and only if $\bigcap_{i\geq 1} D_i(G) = \{1\}$.

This paper deals with combinatorial properties of surface braid groups, in particular, their lower central series, and their related residual properties. In the case of the disc $\mathbb{D}^2$ we have that $B_n(\mathbb{D}^2)$ is residually nilpotent if and only if $n \leq 2$, and

if $n \geq 3$ then $\Gamma_3(B_n(\mathbb{D}^2)) = \Gamma_2(B_n(\mathbb{D}^2))$ (see Proposition 4). Moreover, Gorin and Lin [GL] showed that $\Gamma_2(B_n(\mathbb{D}^2))$ is perfect for $n \geq 5$.

The case of the sphere $\mathbb{S}^2$ and the punctured sphere has been studied by one of the authors and D. Gonçalves [GG2]: in particular $B_n(\mathbb{S}^2)$ is residually nilpotent if and only if $n \leq 2$ and for all $n \geq 3$, $\Gamma_3(B_n(\mathbb{S}^2)) = \Gamma_2(B_n(\mathbb{S}^2))$.

Our main results, which concern orientable surfaces of genus at least one, are as follows.

THEOREM 1 *Let Σ_g be a compact, connected orientable surface without boundary, of genus $g \geq 1$, and let $n \geq 3$. Then:*

(a) $\Gamma_1(B_n(\Sigma_g))/\Gamma_2(B_n(\Sigma_g)) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_2$.

(b) $\Gamma_2(B_n(\Sigma_g))/\Gamma_3(B_n(\Sigma_g)) \cong \mathbb{Z}_{n-1+g}$.

(c) $\Gamma_3(B_n(\Sigma_g)) = \Gamma_4(B_n(\Sigma_g))$. *Moreover* $\Gamma_3(B_n(\Sigma_g))$ *is perfect for* $n \geq 5$.

(d) $B_n(\Sigma_g)$ *is not residually nilpotent.*

This implies that braid groups of compact, connected orientable surfaces without boundary may be distinguished by their lower central series (indeed by the first two lower central quotients).

THEOREM 2 *Let $g \geq 1$, $m \geq 1$ and $n \geq 3$. Let $\Sigma_{g,m}$ be a compact, connected orientable surface of genus g with m boundary components. Then:*

(a) $\Gamma_1(B_n(\Sigma_{g,m}))/\Gamma_2(B_n(\Sigma_{g,m})) = \mathbb{Z}^{2g+m-1} \oplus \mathbb{Z}_2$.

(b) $\Gamma_2(B_n(\Sigma_{g,m}))/\Gamma_3(B_n(\Sigma_{g,m})) = \mathbb{Z}$.

(c) $\Gamma_3(B_n(\Sigma_{g,m})) = \Gamma_4(B_n(\Sigma_{g,m}))$. *Moreover* $\Gamma_3(B_n(\Sigma_{g,m}))$ *is perfect for* $n \geq 5$.

(d) $B_n(\Sigma_{g,m})$ *is not residually nilpotent.*

Braid groups on 2 strands represent a very difficult and interesting case. In the case of the torus, we are able to prove that its 2-strand braid group is residually nilpotent. Further, using ideas from [GG2] and results of [Ga], we may show that apart from the first term, the lower central series of $B_2(\mathbb{T}^2)$ and $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ coincide, and we may also determine all of their successive lower central quotients. More precisely:

THEOREM 3

(a) $B_2(\mathbb{T}^2)$ *is residually nilpotent.*

(b) For all $i \geq 2$:

(i) $\Gamma_i(B_2(\mathbb{T}^2)) \cong \Gamma_i(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2)$.

(ii) $\Gamma_i(B_2(\mathbb{T}^2))/\Gamma_{i+1}(B_2(\mathbb{T}^2))$ *is isomorphic to the direct sum of* R_i *copies of* $\mathbb{Z}_2$*, where:*

$$R_i = \sum_{j=1}^{i-2} \left(\sum_{\substack{k|i-j \\ k>1}} \mu \left(\frac{i-j}{k} \right) \frac{k\alpha_k}{i-j} \right) \quad \textit{and} \quad k\alpha_k = 2^k + 2(-1)^k.$$

(c) $B_2(\mathbb{T}^2)$ *is not residually torsion-free nilpotent.*

Finally, as we shall see in Proposition 14, $B_2(\mathbb{T}^2)$ is not bi-orderable (see Section 2 for a definition).

In Section 5, we recall the relations between mapping class groups and surface braid groups, and prove that pure braid groups of the torus and of surfaces with boundary components are residually torsion-free nilpotent. This is achieved by showing that they may be realised as subgroups of the Torelli group of a surface of higher genus (Lemma 19), which is known to be residually torsion-free nilpotent (see for instance [H]). We note that the embedding proposed in Lemma 19 does not hold when the surface is without boundary (see Remark 20).

Acknowledgements The authors are grateful to I. Marin for pointing out the relevance of the notion of residual torsion-free nilpotence, to V. Bardakov for useful suggestions, and to L. Paris for the reference [H].

2 Lower central series for Artin-Tits groups

Let us start by recalling some standard results on combinatorial properties of braid groups. The following result is well known (see [GL] for instance).

PROPOSITION 4 *Let* B_n *be the Artin braid group on* $n \geq 3$ *strands.*
Then $\Gamma_1(B_n)/\Gamma_2(B_n) \cong \mathbb{Z}$ *and* $\Gamma_2(B_n) = \Gamma_3(B_n)$*.*

Proof. Let us give an easy proof of the second statement (we use an argument of [GG2]). Let $\{\sigma_1, \ldots, \sigma_{n-1}\}$ be the usual set of generators of B_n; the classical relations of B_n, referred to hereafter as *braid relations*, are as follows:

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \text{ for all } 1 \leq i, j \leq n-1 \text{ and } |i-j| \geq 2, \tag{1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } 1 \leq i \leq n-2. \tag{2}$$

From this, we see that $B_n/\Gamma_2(B_n)$ is isomorphic to $\mathbb{Z}$.

Consider the following short exact sequence:

$$1 \longrightarrow \frac{\Gamma_2(B_n)}{\Gamma_3(B_n)} \longrightarrow \frac{B_n}{\Gamma_3(B_n)} \xrightarrow{p} \frac{B_n}{\Gamma_2(B_n)} \longrightarrow 1,$$

Since all of the $\sigma_i \in B_n/\Gamma_3(B_n)$ project to the same element of $B_n/\Gamma_2(B_n)$, for each $1 \leq i \leq n-1$, there exists $t_i \in \Gamma_2(B_n)/\Gamma_3(B_n)$ (with $t_1 = 1$) such that $\sigma_i = t_i\sigma_1$. Projecting the braid relation (2) into $B_n/\Gamma_3(B_n)$, we see that $t_i\sigma_1 t_{i+1}\sigma_1 t_i\sigma_1 = t_{i+1}\sigma_1 t_i\sigma_1 t_{i+1}\sigma_1$. But the t_i are central in $B_n/\Gamma_3(B_n)$, so $t_i = t_{i+1}$, and since $t_1 = 1$, we obtain $\sigma_1 = \ldots = \sigma_{n-1}$. So the surjective homomorphism p is in fact an isomorphism. ∎

We recall that classical braid groups are also called Artin-Tits groups of type $\mathcal{A}$. More precisely, let (W, S) be a Coxeter system and let us denote by $m_{s,t}$ the order of the element st in W (for $s, t \in S$). Let B_W be the group defined by the following group presentation:

$$B_W = \langle S \,|\, \underbrace{st\cdots}_{m_{s,\,t}} = \underbrace{ts\cdots}_{m_{s,\,t}} \;\text{ for any } s \neq t \in S \text{ with } m_{s,t} < +\infty \,\rangle$$

The group B_W is the Artin-Tits group associated to W. The group B_W is said to be of spherical type if W is finite. The kernel of the canonical projection of B_W onto W is called the pure Artin-Tits group associated to W.

Proposition 5 *Let B_W be an Artin-Tits group of spherical type, with W different from the dihedral group I_{2m}. Then:*

i) $\Gamma_1(B_W)/\Gamma_2(B_W)$ *is isomorphic to* $\mathbb{Z}$ *or* $\mathbb{Z}^2$.

ii) $\Gamma_2(B_W) = \Gamma_3(B_W)$.

Proof. Considering B_W equipped with the previous group presentation, it is easy to calculate the Abelianisation. Using the same argument as in Proposition 4, one deduces that if s and t are two generators of an Artin-Tits group of spherical type B_W then they are identified in $B_W/\Gamma_3(B_W)$ if $m_{s,t}$ is odd. This argument allows us to prove that $B_W/\Gamma_3(B_W)$ is isomorphic to $B_W/\Gamma_2(B_W)$ for almost all Artin-Tits groups of spherical type, the only exception being that of the one-relator group $B_{I_{2m}} = \langle a, b \,|\, (ab)^m = (ba)^m \rangle$ (for $m > 1$), since the defining relation is of even length. ∎

Remarks 6 The Artin-Tits group $B_{I_{2m}} = \langle a, b \,|\, (ab)^m = (ba)^m \rangle$ is residually nilpotent if and only if m is a power of a prime number. Indeed, by taking $c = ba$, it is readily seen that the group B_W is isomorphic to the Baumslag-Solitar group of type (m, m), $BS_m = \langle a, c \,|\, [a, c^m] = 1 \rangle$, which is known to be residually nilpotent if m is a power of a prime number. Conversely, let G be a one-relator group with non-trivial centre. According to [McC], G is residually nilpotent if and only if one of the following holds:

i) G is Abelian.

ii) G is isomorphic to a Baumslag-Solitar group of type (r,r), with r a power of a prime number.

iii) G is isomorphic to $G_{p,q} = \langle m, n \,|\, m^p = n^q \rangle$ where p and q are powers of the same prime number.

Suppose that $B_{I_{2m}}$ is residually nilpotent. The group $B_{I_{2m}}$ is not Abelian for $m > 1$, and it cannot be isomorphic to a group $G_{p,q}$ since they have different Abelianisations. Therefore $B_{I_{2m}}$ is isomorphic to a Baumslag-Solitar group of type (r,r) with r a power of a prime number. But as we remarked, $B_{I_{2m}}$ is also isomorphic to the Baumslag-Solitar group of type (m,m), in which case $m = r$, by a result of Moldavanskii on isomorphisms of Baumslag-Solitar groups ([Mol]). It thus follows that $B_{I_{2m}}$ is residually nilpotent if and only if m is a power of a prime number.

On the other hand, it is well known that pure braid groups are residually torsion-free nilpotent [FR]. Using the faithfulness of the Krammer-Digne representation, Marin has shown recently that the pure Artin-Tits groups of spherical type are residually torsion-free nilpotent [M].

The fact that a group is residually torsion-free nilpotent has several important consequences, notably that the group is bi-orderable [MR]. We recall that a group G is said to be *bi-orderable* if there exists a strict total ordering $<$ on its elements which is invariant under left and right multiplication, in other words, $g < h$ implies that $gk < hk$ and $kg < kh$ for all $g, h, k \in G$. We state one interesting property of bi-orderable groups. A group G is said to have *generalised torsion* if there exist $g, h_1, \ldots, h_k$, $(g \neq 1)$ such that:

$$(h_1 g h_1^{-1})(h_2 g h_2^{-1}) \cdots (h_k g h_k^{-1}) = 1 \,.$$

PROPOSITION 7 ([KK]) *A bi-orderable group has no generalised torsion.*

The braid group B_n is not bi-orderable for $n \geq 3$ since it has generalised torsion (see [N] or [Ba]). As we shall see in Section 4, $B_2(\mathbb{T}^2)$ *is* residually nilpotent, but is not bi-orderable.

3 Lower central series for surface braid groups on at least 3 strands

3.1 Surfaces without boundary

This section is devoted to proving Theorem 1. Let Σ_g be a compact, connected orientable surface without boundary, of genus $g > 0$. We start by giving a presentation of $B_n(\Sigma_g)$.

THEOREM 8 *([B]) Let $n \in \mathbb{N}$. Then $B_n(\Sigma_g)$ admits the following group presentation:*

Generators: $a_1, b_1, \ldots, a_g, b_g, \sigma_1, \ldots, \sigma_{n-1}$.

Relations:

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \tag{3}$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for all } 1 \leq i \leq n-2 \tag{4}$$
$$c_i\sigma_j = \sigma_j c_i \text{ for all } j \geq 2,\ c_i = a_i \text{ or } b_i \text{ and } i = 1, \ldots, g \tag{5}$$
$$c_i\sigma_1 c_i\sigma_1 = \sigma_1 c_i\sigma_1 c_i \text{ for } c_i = a_i \text{ or } b_i \text{ and } i = 1, \ldots, g \tag{6}$$
$$a_i\sigma_1 b_i = \sigma_1 b_i\sigma_1 a_i\sigma_1 \text{ for } i = 1, \ldots, g \tag{7}$$
$$c_i\sigma_1^{-1}c_j\sigma_1 = \sigma_1^{-1}c_j\sigma_1 c_i \text{ for } c_i = a_i \text{ or } b_i,\ c_j = a_j \text{ or } b_j \text{ and } 1 \leq j < i \leq g \tag{8}$$
$$\prod_{i=1}^{g}[a_i^{-1}, b_i] = \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\cdots\sigma_1. \tag{9}$$

Proof. Let $\widetilde{B_n(\Sigma_g)}$ be the group defined by the above presentation, and let $B_n(\Sigma_g)$ be the group given by the presentation of Theorem 1.2 of [B]. Consider the homomorphism $\varphi\colon\ B_n(\Sigma_g) \longrightarrow \widetilde{B_n(\Sigma_g)}$ defined on the generators of $B_n(\Sigma_g)$ by $\varphi(\sigma_j) = \sigma_j$ (for $j = 1, \ldots, n-1$), $\varphi(a_i) = a_i^{-1}$ and $\varphi(b_i) = b_i^{-1}$ (for $i = 1, \ldots, g$). It is an easy exercise to check that φ is an isomorphism. ∎

Proof of Theorem 1.

(a) Consider the group $\mathbb{Z}^{2g} \oplus \mathbb{Z}_2$ defined by the presentation $\langle c_1, \ldots, c_{2g}, \sigma \,|\, \sigma^2 = [c_i, c_j] = [c_i, \sigma] = 1,\ \text{for}\ 1 \leq i, j \leq 2g \rangle$ and $B_n(\Sigma_g)$ with the group presentation given by Theorem 8. It is easy to check that the homomorphism

$$\varphi\colon\ \Gamma_1(B_n(\Sigma_g))/\Gamma_2(B_n(\Sigma_g)) \longrightarrow \mathbb{Z}^{2g} \oplus \mathbb{Z}_2$$

which sends a_k to c_{2k-1}, b_k to c_{2k} and every σ_j to σ is indeed an isomorphism.

(b) Let us start by determining a group presentation for $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$. Let q be the canonical projection of $B_n(\Sigma_g)$ onto $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$. As in the proof of Proposition 4, the braid relations (4) imply that $q(\sigma_1) = \cdots = q(\sigma_{n-1})$; we denote this element by σ. This implies that the projected relations (3) are trivial. For $i = 1, \ldots, g$, let us also denote $q(a_i)$ by a_i and $q(b_i)$ by b_i. Since $n \geq 3$, we see from relations (5) that σ is central in $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ and hence the projected relations (6) become trivial. From relations (8), for all $1 \leq i, j \leq g$, $i \neq j$, one may infer that $[a_i, b_j] = [a_i, a_j] = [b_i, b_j] = [b_i, a_j] = 1$ in $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$. Relations (7) and (9) imply that $[b_i, a_i] = \sigma^{-2}$ for all $= 1, \ldots, g$, and $\prod_{i=1}^{g}[a_i^{-1}, b_i] = \sigma^{2(n-1)}$ respectively. Conjugating the latter equation by $a_1 \cdots a_g$ yields $\prod_{i=1}^{g}[b_i, a_i] = \sigma^{2(n-1)}$ in $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ (recall that a_i commutes with a_j and b_j if $i \neq j$), and

hence $\sigma^{-2g} = \prod_{i=1}^{g} [b_i, a_i] = \sigma^{2(n-1)}$. Therefore, $\sigma^{2(g+n-1)} = 1$ in $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$. Summing up, we have obtained the following information:

$$\left.\begin{array}{l} B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g)) \text{ is generated by } a_1, b_1, \ldots, a_g, b_g \text{ and } \sigma \\ a_1, b_1, \ldots, a_g, b_g \text{ and } \sigma \text{ commute pairwise except for the pairs } (a_i, b_i)_{i=1,\ldots,g} \\ [a_1, b_1] = \cdots = [a_g, b_g] = \sigma^2; \qquad \sigma^{2(n+g-1)} = 1. \end{array}\right\} \quad (10)$$

The remaining relations of $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ are those of the form $[[x, y], z] = 1$ for all $x, y, z \in B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$. We claim that such relations are implied by those of (10). To see this, recall that $\Gamma_2(B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g)))$ is the normal subgroup of $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ generated by the finite set of commutators $[a_i, a_j]$, $[b_i, b_j]$, $[a_i, b_j]$, $[a_i, b_i]$, $[a_i, \sigma]$ and $[b_i, \sigma]$, for $1 \le i \ne j \le g$. But the relations of (10) imply that these commutators are all trivial, with the exception of $[a_i, b_i]$ for $1 \le i \le g$, which is equal to σ^2. Since σ is central in $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$, we conclude that $\Gamma_2(B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))) = \langle \sigma^2 \rangle$, and that $[[x, y], z] = 1$ as claimed. Hence (10) is a group presentation for $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$.

Now consider the following exact sequence:

$$1 \longrightarrow \frac{\Gamma_2(B_n(\Sigma_g))}{\Gamma_3(B_n(\Sigma_g))} \longrightarrow \frac{B_n(\Sigma_g)}{\Gamma_3(B_n(\Sigma_g))} \xrightarrow{p} \frac{B_n(\Sigma_g)}{\Gamma_2(B_n(\Sigma_g)} \longrightarrow 1.$$

From the presentation of $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ given by (10), one sees that every element x of $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ may be written in the form $a_1^{j_1} b_1^{k_1} \cdots a_g^{j_g} b_g^{k_g} \sigma^p$. Since $B_n(\Sigma_g)/\Gamma_2(B_n(\Sigma_g))$ is isomorphic to $\mathbb{Z}^{2g} \oplus \mathbb{Z}_2$, the factors being generated respectively by $p(a_1), p(b_1), \ldots, p(a_g), p(b_g)$ and $p(\sigma)$, if $x \in \mathrm{Ker}(p)$ then $j_1 = k_1 = \ldots = j_g = k_g = 0$ and p is even, so $\mathrm{Ker}(p) \subseteq \langle \sigma^2 \rangle$. The converse is clearly true and so $\mathrm{Ker}(p) = \langle \sigma^2 \rangle$.

Let d denote the order of σ^2 in $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$. From (10), we have that d divides $n + g - 1$. To complete the proof of part (b) of Theorem 1, it suffices to show that $n + g - 1$ divides d.

Let G be the group generated by elements $a_1, b_1, \ldots, a_g, b_g$ and σ, whose relations are $\sigma^{2(n+g-1)} = 1$, and the generators commute pairwise except for the pairs (a_i, b_i) for $i = 1, \ldots, g$. Then $G = (\oplus_{i=1}^{g} \mathbb{F}_2(a_i, b_i)) \oplus \mathbb{Z}_{2(n+g-1)}$, where $\mathbb{F}_2(a_i, b_i)$ denotes the free group of rank 2 generated by a_i and b_i. Let N be the subgroup of G normally generated by the elements $[a_1, b_1]\sigma^{-2}, \ldots, [a_g, b_g]\sigma^{-2}$, and let ρ denote the canonical projection $G \longrightarrow G/N$. Then $G/N \cong B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ by the group presentation given in (10). The cosets modulo N of the elements $a_1, b_1, \ldots, a_g, b_g$ and σ of G may be identified respectively with the elements $a_1, b_1, \ldots, a_g, b_g$ and σ of $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$. Further, by applying the relations of G, any element of N may be written in the form $\prod_{i=1}^{g} \left(\prod_{k=1}^{m_i} \, u_{i_k} [a_i, b_i]^{\varepsilon_{i_k}} u_{i_k}^{-1} \right) \sigma^{-2\left(\sum_{i=1}^{g} \left(\sum_{k=1}^{m_i} \, \varepsilon_{i_k} \right) \right)}$, where $m_i \in \mathbb{N}$ for all $i = 1, \ldots, g$, and for all $k = 1, \ldots, m_i$, $u_{i_k} \in \mathbb{F}_2(a_i, b_i)$ and $\varepsilon_{i_k} \in \{1, -1\}$. Since $\sigma^{2d} = 1$ in $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$, and so in G/N, considered as an

element of G, it follows that σ^{2d} belongs to $\mathrm{Ker}(\rho)$. Hence for all $i = 1, \ldots, g$, there exists $m_i \in \mathbb{N}$, and for $1 \leq k \leq m_i$, there exist $u_{i_k} \in \mathbb{F}_2(a_i, b_i)$ and $\varepsilon_{i_k} \in \{1, -1\}$ such that $\sigma^{2d} = \prod_{i=1}^{g} \left(\prod_{k=1}^{m_i} u_{i_k} [a_i, b_i]^{\varepsilon_{i_k}} u_{i_k}^{-1} \right) \sigma^{-2\left(\sum_{i=1}^{g}\left(\sum_{k=1}^{m_i} \varepsilon_{i_k}\right)\right)}$. Thus:

$$\sigma^{2\left(d+\sum_{i=1}^{g}\left(\sum_{k=1}^{m_i} \varepsilon_{i_k}\right)\right)} = \prod_{i=1}^{g} \left(\prod_{k=1}^{m_i} u_{i_k} ([a_i, b_i])^{\varepsilon_{i_k}} u_{i_k}^{-1} \right). \tag{11}$$

From the structure of G, it follows that both the right- and left-hand sides are equal to 1. Moreover, $\Gamma_2(G) = \oplus_{i=1}^{g} \Gamma_2(\mathbb{F}_2(a_i, b_i))$. Let $1 \leq i \leq g$. Projecting the right-hand side of equation (11), which belongs to $\Gamma_2(G)$, into $\Gamma_2(\mathbb{F}_2(a_i, b_i))$, and then into $\Gamma_2(\mathbb{F}_2(a_i, b_i))/\Gamma_3(\mathbb{F}_2(a_i, b_i))$, we observe that $[a_i, b_i]^{\sum_{k=1}^{m_i} \varepsilon_{i_k}} = 1$. But this quotient is an infinite cyclic group [MKS], hence $\sum_{k=1}^{m_i} \varepsilon_{i_k} = 0$ for $i = 1, \ldots, g$ and therefore $\sum_{i=1}^{g} \left(\sum_{k=1}^{m_i} \varepsilon_{i_k} \right) = 0$. Thus the left-hand side of equation (11) reduces to $\sigma^{2d} = 1$ in G, and so $n + g - 1$ divides d. It follows that σ is of order $2(n + g - 1)$ in $B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ as claimed.

(c) Let H denote the normal closure in $B_n(\Sigma_g)$ of the element $\sigma_1 \sigma_2^{-1}$. Using the Artin braid relations, one may check that in $B_n(\Sigma_g)/H$, the σ_i are all identified to a single element, σ, say, and then that equation (10) defines a group presentation for $B_n(\Sigma_g)/H$. Thus $B_n(\Sigma_g)/H \cong B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g))$ via an isomorphism ι. Now $B_n(\Sigma_g)$ contains a copy of the usual Artin braid group B_n which is generated by the σ_i. From the Artin braid relations, it follows that $\Gamma_2(B_n)$ is the normal closure in B_n of the elements $\sigma_i \sigma_{i+1}^{-1}$, $1 \leq i \leq n-2$. Moreover, since $\sigma_{i+1}\sigma_{i+2}^{-1} = \sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_{i+2}^{-1} \cdot \sigma_i \sigma_{i+1}^{-1} \cdot \sigma_{i+2}\sigma_{i+1}\sigma_i$ for all $1 \leq i \leq n-3$, we see that $\Gamma_2(B_n)$ is the normal closure in B_n of just $\sigma_1\sigma_2^{-1}$, and thus $\mathcal{N}_{B_n(\Sigma_g)}(\Gamma_2(B_n)) = H$ (if X is a subset of a group G, then we denote its normal closure in G by $\mathcal{N}_G(X)$).

Since $\Gamma_3(B_n) = \Gamma_2(B_n)$ (by Proposition 4), we have $\Gamma_4(B_n(\Sigma_g)) \supseteq \Gamma_4(B_n) = \Gamma_2(B_n)$. Taking normal closures in $B_n(\Sigma_g)$, we deduce that H is a normal subgroup of $\Gamma_4(B_n(\Sigma_g))$, and hence we obtain the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \Gamma_4(B_n(\Sigma_g))/H \arrow[r] & B_n(\Sigma_g)/H \arrow[r, two heads] \arrow[dr, "\iota", "\cong"'] & B_n(\Sigma_g)/\Gamma_4(B_n(\Sigma_g)) \arrow[r] \arrow[d] & 1 \\
& & & B_n(\Sigma_g)/\Gamma_3(B_n(\Sigma_g)). &
\end{tikzcd}$$

Since ι is an isomorphism, so is the vertical arrow, and hence its kernel $\Gamma_3(B_n(\Sigma_g))/\Gamma_4(B_n(\Sigma_g))$ is trivial. This proves the first part of (c). To prove the second part, we have just seen that the normal closure H of $\Gamma_2(B_n)$ in $B_n(\Sigma_g)$, is isomorphic to $\Gamma_3(B_n(\Sigma_g))$ (they coincide in fact). Since $\Gamma_2(B_n)$ is perfect for all $n \geq 5$ [GL], so are H and $\Gamma_3(B_n(\Sigma_g))$.

(d) We first remark that $\Gamma_3(B_n(\Sigma_g)) \neq \{1\}$. For if $\Gamma_3(B_n(\Sigma_g))$ were trivial, by (c), we would have $\Gamma_2(B_n(\Sigma_g)) \cong \mathbb{Z}_{n-1+g}$. But by [VB], $B_n(\Sigma_g)$ is torsion free, which

yields a contradiction. From this it follows that $\bigcap_{i\in\mathbb{N}} \Gamma_i(B_n(\Sigma_g)) \neq \{1\}$. This completes the proof of the theorem.

∎

REMARK 9 Given a group G, the property that the ith term $\Gamma_i(G)$ is perfect implies that $\Gamma_i(G) = \Gamma_{i+1}(G)$.

3.2 Surfaces with non-empty boundary

In this section, we study the case of orientable surfaces with boundary, and prove Theorem 2. We identify $\Sigma_{g,0}$ with Σ_g. As in Theorem 8, from Theorem 1.1 of [B], one obtains the following presentation of $B_n(\Sigma_{g,m})$.

THEOREM 10 *Let $n \in \mathbb{N}$. Then $B_n(\Sigma_{g,m})$ admits the following group presentation:*

Generators: $a_1, b_1, \ldots, a_g, b_g, z_1, \ldots z_{m-1}, \sigma_1, \ldots, \sigma_{n-1}$.

Relations:

Relations (3) – (8) of Theorem 8

$$z_i\sigma_j = \sigma_j z_i \text{ for all } j \geq 2 \text{ and } i = 1, \ldots, m-1 \tag{12}$$

$$z_i\sigma_1 z_i\sigma_1 = \sigma_1 z_i\sigma_1 z_i \text{ for } i = 1, \ldots, m-1 \tag{13}$$

$$z_i\sigma_1^{-1} z_j\sigma_1 = \sigma_1^{-1} z_j\sigma_1 z_i \text{ for } 1 \leq j < i \leq m-1 \tag{14}$$

$$c_i\sigma_1^{-1} z_j\sigma_1 = \sigma_1^{-1} z_j\sigma_1 c_i \text{ for } c_i = a_i \text{ or } b_i,\ i = 1, \ldots, g \text{ and } j = 1, \ldots, m-1. \tag{15}$$

Proof of Theorem 2. Statement (a) may be proved in the same way as (a) of Theorem 1.

We now prove part (b). As in the proof of part (b) of Theorem 1, one may check that $\Gamma_2(B_n(\Sigma_{g,m}))/\Gamma_3(B_n(\Sigma_{g,m})) = \langle \sigma^2 \rangle$, where for all $i = 1, \ldots, n-1$, σ is the projection of σ_i in $B_n(\Sigma_{g,m})/\Gamma_3(B_n(\Sigma_{g,m}))$. It thus suffices to show that σ^2 is of infinite order.

Instead of repeating the arguments used in Theorem 1, we propose a different proof, based on geometric relations between surface braid groups. Suppose that $\sigma^{2d} = 1$ for some $d \in \mathbb{N}$. This is equivalent to saying that σ_i^{2d} belongs to $\Gamma_3(B_n(\Sigma_{g,m}))$ for all $i = 1, \ldots, n-1$.

Let $1 \leq i \leq m$. To each boundary component ∂_i of $\Sigma_{g,m}$ let us associate a surface $\Sigma_{g_i,1}$ of positive genus g_i. We choose the g_i so that $h = g + \sum_{i=1}^{m} g_i > d - (n-1)$. Let Σ_h denote the compact, orientable surface without boundary and of genus h obtained by glueing $\partial\Sigma_{g_i,1}$ to ∂_i for all $i = 1, \ldots, m$. The embedding of $\Sigma_{g,m}$ into Σ_h induces a natural homomorphism φ between $B_n(\Sigma_{g,m})$ and $B_n(\Sigma_h)$,

sending geometric generators of $B_n(\Sigma_{g,m})$ to the corresponding elements of $B_n(\Sigma_h)$. In particular, $\varphi(\sigma_i) = \sigma_i$ for all $i = 1, \ldots, n-1$.

Since σ_i^{2d} belongs to $\Gamma_3(B_n(\Sigma_{g,m}))$, it follows that $\varphi(\sigma_i^{2d}) = \sigma_i^{2d}$ belongs to $\Gamma_3(B_n(\Sigma_h))$, and hence $\sigma^{2d} = 1$ in $\Gamma_2(B_n(\Sigma_h))/\Gamma_3(B_n(\Sigma_h))$ (recall that, by Theorem 1, $\Gamma_2(B_n(\Sigma_h))/\Gamma_3(B_n(\Sigma_h)) = \langle \sigma^2 \rangle \cong \mathbb{Z}_{h+n-1}$). But this would imply that $h + n - 1 \leq d$ – a contradiction. This proves part (b).

Part (c) may be proved in the same way as (c) of Theorem 1; indeed, the quotient $B_n(\Sigma_{g,m})/\Gamma_3(B_n(\Sigma_{g,m}))$ has a presentation similar to that of (10) (with the z_i central, but without the last relation), and is isomorphic to $B_n(\Sigma_{g,m})/H$, where H is the normal closure of $\sigma_1\sigma_2^{-1}$ in $B_n(\Sigma_{g,m})$, and thus is equal to the normal closure of $\Gamma_2(B_n)$ in $B_n(\Sigma_{g,m})$. As in Theorem 1, one may show that $H = \Gamma_3(B_n(\Sigma_{g,m})$ is perfect for $n \geq 5$.

Finally, to prove part (d), as in Theorem 1 it suffices to prove that $\Gamma_3(B_n(\Sigma_{g,m})) \neq \{1\}$. Suppose that $\Gamma_3(B_n(\Sigma_{g,m})) = \{1\}$. Then $\Gamma_2(B_n(\Sigma_{g,m})) \cong \mathbb{Z}$ by (b), and since $B_n(\Sigma_{g,m}) \supset B_n$, it follows that $\Gamma_2(B_n)$ is cyclic; but since $n \geq 3$, this contradicts the results of [GL]. ∎

4 Braid groups on 2 strands: properties and open questions

The aim of this section is to prove Theorem 3. Consider first the group presentation given by Theorem 8, and take $n = 2$ and $g = 1$. Setting $\alpha = a\sigma_1$, $\beta = b\sigma_1$ and $\gamma = a\sigma_1 b$, one obtains the following presentation of $B_2(\mathbb{T}^2)$:

THEOREM 11 ([BG]) *$B_2(\mathbb{T}^2)$ is generated by α, β and γ, subject to the relations:*

$$\alpha^2 \text{ and } \beta^2 \text{ are central}$$
$$\alpha^2\beta^2 = \gamma^2.$$

Further, α^2 and β^2 generate the centre of $B_2(\mathbb{T}^2)$.

Let $p\colon\ B_2(\mathbb{T}^2) \longrightarrow B_2(\mathbb{T}^2)/Z(B_2(\mathbb{T}^2))$ denote the canonical projection. From this presentation, it follows that $B_2(\mathbb{T}^2)/Z(B_2(\mathbb{T}^2))$ is generated by $\overline{\alpha} = p(\alpha)$, $\overline{\beta} = p(\beta)$ and $\overline{\gamma} = p(\gamma)$, subject to the relations $\overline{\alpha}^2 = \overline{\beta}^2 = \overline{\gamma}^2 = 1$. So $B_2(\mathbb{T}^2)/Z(B_2(\mathbb{T}^2))$, which we identify with $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, is the Coxeter group $W(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ associated to the free group $\mathbb{F}_3(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$, and $B_2(\mathbb{T}^2)$ is a central extension of $W(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$:

$$1 \longrightarrow Z(B_2(\mathbb{T}^2)) \longrightarrow B_2(\mathbb{T}^2) \xrightarrow{p} \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow 1.$$

This presentation of $B_2(\mathbb{T}^2)$ was considered in [BG], where the following length functions $\ell_{\widehat{\alpha}}, \ell_{\widehat{\beta}}$ were defined. If $x \in \{\alpha, \beta, \gamma\}$, set:

$$\ell_{\widehat{\alpha}}(x) = \begin{cases} 1 \text{ if } x \neq \alpha \\ 0 \text{ if } x = \alpha, \end{cases}$$

and similarly for $\ell_{\widehat{\beta}}$. From Theorem 11, it follows that each of $\ell_{\widehat{\alpha}}$ and $\ell_{\widehat{\beta}}$ extends to a homomorphism of $B_2(\mathbb{T}^2)$ onto $\mathbb{Z}$.

The following observation will be used in the proof of Theorem 3.

PROPOSITION 12 *The intersection of* $\Gamma_2(B_2(\mathbb{T}^2))$ *and* $Z(B_2(\mathbb{T}^2))$ *is trivial.*

Proof. Let $x \in Z(B_2(\mathbb{T}^2))$. By Theorem 11, there exist $m, n \in \mathbb{Z}$ such that $x = a^{2m}b^{2n}$, and thus $\ell_{\widehat{\alpha}}(x) = 2n$ and $\ell_{\widehat{\beta}}(x) = 2m$. But $x \in \Gamma_2(B_2(\mathbb{T}^2))$, so $\ell_{\widehat{\alpha}}(x) = \ell_{\widehat{\beta}}(x) = 0$. We conclude that $m = n = 0$, and hence $x = 1$. ∎

We are now able to prove Theorem 3.

Proof of Theorem 3. Set $G = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$.

(a) Suppose that $x \in \bigcap_{i\in\mathbb{N}} \Gamma_i(B_2(\mathbb{T}^2))$. Then $p(x) \in \bigcap_{i\in\mathbb{N}} \Gamma_i(G)$, but since G is residually nilpotent [G], it follows that $x \in \mathrm{Ker}(p) = Z(B_2(\mathbb{T}^2))$. So $x = 1$ by Proposition 12, and hence $B_2(\mathbb{T}^2)$ is residually nilpotent.

(b) *(i)* Let us consider the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & \Gamma_2(B_2(\mathbb{T}^2)) \arrow[r] \arrow[d, "p_2"'] & B_2(\mathbb{T}^2) \arrow[r] \arrow[d, "p"'] & B_n(\mathbb{T}^2)/\Gamma_2(B_n(\mathbb{T}^2)) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \Gamma_2(G) \arrow[r] & G \arrow[r] & G/\Gamma_2(G) \arrow[r] & 1
\end{tikzcd}$$

The first and third vertical arrows are those induced by p. More generally, for $i \geq 2$, let $p_i \colon \Gamma_i(B_2(\mathbb{T}^2)) \longrightarrow \Gamma_i(G)$ denote the epimorphism induced by p. But it follows from Proposition 12 that p_2 is also injective, so is an isomorphism. Since for $i \geq 3$, p_i is the restriction of p_2 to $\Gamma_i(B_2(\mathbb{T}^2))$, p_i is an isomorphism too.

(ii) From (b)(i), it follows that $\Gamma_i(B_2(\mathbb{T}^2))/\Gamma_{i+1}(B_2(\mathbb{T}^2)) \cong \Gamma_i(G)/\Gamma_{i+1}(G)$, so it suffices to prove the result for G. We break the proof down into two parts as follows:

(1). Recall that the elements $\overline{\alpha}, \overline{\beta}$ and $\overline{\gamma}$ are each of order 2, and generate G. We claim that every non-trivial element of $\Gamma_i(G)/\Gamma_{i+1}(G)$ is of order 2. Since $\Gamma_i(G)/\Gamma_{i+1}(G)$ is a finitely-generated Abelian group by [MKS], this will imply that it is isomorphic to a finite number, R_i say, of copies of $\mathbb{Z}_2$. To prove the claim, recall from [MKS] that $\Gamma_i(G)/\Gamma_{i+1}(G)$ is generated by the cosets modulo $\Gamma_{i+1}(G)$ of the i-fold simple commutators $[[\cdots[[\rho_1, \rho_2], \rho_3]\cdots, \rho_{i-1}], \rho_i]$, where $\rho_j \in \left\{\overline{\alpha}, \overline{\beta}, \overline{\gamma}\right\}$ for all

$1 \leq j \leq i$. We argue by induction on $i \geq 2$. Firstly, let $i = 2$. Then $\Gamma_2(G)/\Gamma_3(G)$ is generated by the cosets of the $[\rho_1, \rho_2]$. But modulo $\Gamma_3(G)$, $[\rho_1, \rho_2]^2 \equiv [\rho_1^2, \rho_2] \equiv 1$, and since $\Gamma_2(G)/\Gamma_3(G)$ is Abelian, this implies that all of its non-trivial elements are of order 2. Now suppose that $i \geq 3$, and suppose by induction that the result holds for $i-1$, so that $x^2 \equiv 1$ modulo $\Gamma_i(G)$ for all $x \in \Gamma_{i-1}(G)$. Every i-fold simple commutator may be written in the form $[x, \rho_i]$, where x is a $(i-1)$-fold simple commutator, so belongs to $\Gamma_{i-1}(G)$, and $\rho_i \in \left\{\overline{\alpha}, \overline{\beta}, \overline{\gamma}\right\}$, so belongs to G. By the induction hypothesis, $x^2 \in \Gamma_i(G)$, so $[x, \rho_i]^2 \equiv [x^2, \rho_i] \equiv 1$ modulo $\Gamma_{i+1}(G)$, and once more, since $\Gamma_i(G)/\Gamma_{i+1}(G)$ is Abelian, all of its non-trivial elements are of order 2. This proves the claim.

(2). The number R_i of summands of $\mathbb{Z}_2$ is given by Theorem 3.4 of [Ga]. We refer to Gaglione's notation in what follows. Since $U_\infty(x) = 0$, the R_∞^j are all zero (R_∞^j represents the rank of the free Abelian factor of $\Gamma_j(G)/\Gamma_{j+1}(G)$), and so R_i is as given in the statement of the theorem. It just remains to determine $k\alpha_k$ for all $k \geq 2$. A simple calculation shows that $1 - U(x) = (1+x)^2(1-2x)$, hence:

$$\frac{d}{dx} \ln(1 - U(x)) = \frac{2}{x+1} + \frac{2}{2x-1},$$

and that for $k \geq 2$,

$$\frac{d^k}{dx^k} \ln(1 - U(x)) = (-1)^{k+1}(k-1)! \left(\frac{2}{(x+1)^k} + \frac{2^k}{(2x-1)^k} \right).$$

So

$$k\alpha_k = -\frac{1}{(k-1)!} \left(\frac{d^k}{dx^k} \ln(1 - U(x)) \right)\bigg|_{x=0} = 2^k + 2(-1)^k,$$

as required.

(c) Given a group G, the quotient group $D_i(G)/D_{i+1}(G)$ is torsion free and it is isomorphic to $\Gamma_i(G)/\Gamma_{i+1}(G)$ modulo torsion, for $i \geq 1$ [P]. Therefore, from part (b) one deduces that $D_2(B_2(\mathbb{T}^2)) = D_3(B_2(\mathbb{T}^2))$. On the other hand one can easily verify that $B_2(\mathbb{T}^2)/\Gamma_2(B_2(\mathbb{T}^2)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}_2$ and therefore $B_2(\mathbb{T}^2)/D_2(B_2(\mathbb{T}^2)) \cong \mathbb{Z}^2$. Since $B_2(\mathbb{T}^2)$ is not Abelian, it follows that $D_2(B_2(\mathbb{T}^2))$ is not trivial and then $\bigcap_{i \in \mathbb{N}} D_i(B_2(\mathbb{T}^2) \neq \{1\}$.

∎

REMARKS 13 From Theorem 3 one concludes that $\Gamma_i(B_2(\Sigma_{1,p})) \neq \Gamma_{i+1}(B_2(\Sigma_{1,p}))$. On the other hand, the group $\Gamma_2(B_2(\Sigma_{g,p}))$ is generated by the set of conjugates of the commutators of the form $[g, g']$, where g, g' are generators of $B_2(\Sigma_{g,p})$. Therefore $\Gamma_2(B_2(\Sigma_{g,p})) \subset P_2(\Sigma_{g,p})$. Since $P_2(\Sigma_{g,p})$ is residually nilpotent for $p \geq 1$ (see Section 5), one deduces that $B_2(\Sigma_{g,p})$ is residually soluble for $p \geq 1$. The question of whether $B_2(\Sigma)$ is in fact residually nilpotent, when Σ is a surface of positive genus possibly with boundary different from the torus, is open.

To finish this section, we prove the following result:

PROPOSITION 14 *The group $B_2(\mathbb{T}^2)$ is not bi-orderable.*

Proof. Consider $B_2(\mathbb{T}^2)$ with the group presentation in Theorem 11. Set $g = \alpha\beta\gamma^{-1}$. The following equality holds in $B_2(\mathbb{T}^2)$:

$$((\alpha\gamma)^{-1}g(\alpha\gamma))(\gamma^{-1}g\gamma)(\alpha^{-1}g\alpha)(g) = 1\,.$$

Since $g \neq 1$, the group $B_2(\mathbb{T}^2)$ is not bi-orderable by Proposition 7. ∎

Let Σ be an orientable surface, possibly with boundary. If $n \geq 3$, $B_n(\Sigma)$ is not bi-orderable since it contains a copy of B_n which is not bi-orderable [Go]. If $n = 1$, the group $B_1(\Sigma)$ is isomorphic to $\pi_1(\Sigma)$ which is known to be residually free. Therefore it is also residually torsion-free nilpotent and hence bi-orderable.

REMARK 15 If Σ is an orientable surface, possibly with boundary, different from the torus, the sphere and the disc, the question of whether $B_2(\Sigma)$ is in fact bi-orderable is open.

5 Residual torsion free nilpotence of surface pure braid groups

In this section we give a short survey on relations between surface braids and mapping classes, and we show that pure braid groups of surfaces with non-empty boundary may be realised as subgroups of Torelli groups of surfaces with one boundary component.

5.1 Surface pure braid groups

We start by recalling a group presentation for pure braid groups of surfaces with one boundary component [B].

THEOREM 16 *Let $\Sigma_{g,1}$ be a compact, connected orientable surface of genus $g \geq 1$ with one boundary component. The group $P_n(\Sigma_{g,1})$ admits the following presentation:*

Generators: $\{A_{i,j} \mid 1 \leq i \leq 2g+n-1, 2g+1 \leq j \leq 2g+n, i < j\}$.

Relations:

(PR1) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{r,s}$ *if* $(i < j < r < s)$ *or* $(r+1 < i < j < s)$,
or $(i = r+1 < j < s$ *for even* $r < 2g$ *or* $r > 2g)$;

(PR2) $A_{i,j}^{-1} A_{j,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1}$ *if* $(i < j < s)$;

(PR3) $A_{i,j}^{-1} A_{i,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$ *if* $(i < j < s)$;

(PR4) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1} A_{j,s}^{-1} A_{r,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$
if $(i+1 < r < j < s)$ *or*
$(i+1 = r < j < s$ *for odd* $r < 2g$ *or* $r > 2g)$;

(ER1) $A_{r+1,j}^{-1} A_{r,s} A_{r+1,j} = A_{r,s} A_{r+1,s} A_{j,s}^{-1} A_{r+1,s}^{-1}$
if r *odd and* $r < 2g$;

(ER2) $A_{r-1,j}^{-1} A_{r,s} A_{r-1,j} = A_{r-1,s} A_{j,s} A_{r-1,s}^{-1} A_{r,s} A_{j,s} A_{r-1,s} A_{j,s}^{-1} A_{r-1,s}^{-1}$
if r *even and* $r < 2g$.

As a representative of the generator $A_{i,j}$, we may take a geometric braid whose only non-trivial (non-vertical) strand is the $(j-2g)$th one. In Figure 1, we illustrate the projection of such braids on the surface $\Sigma_{g,1}$ (see also Figure 8 of [B]). Some misprints in Relations (ER1) and (ER2) of Theorem 5.1 of [B] have been corrected.

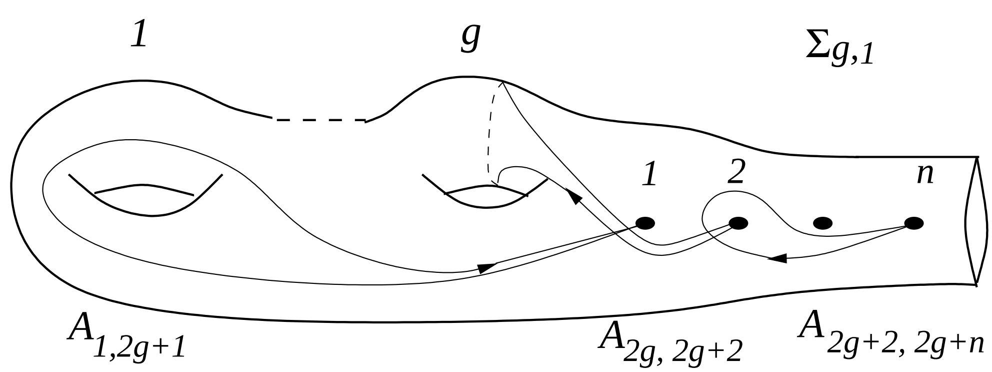

Figure 1: Projection of representatives of the generators $A_{i,j}$. We represent $A_{i,j}$ by its only non-trivial strand.

With respect to the presentation of $B_n(\Sigma_g)$ given in Theorem 8, the elements $A_{i,j}$ are the following braids:

- $A_{i,j} = \sigma_{j-2g} \cdots \sigma_{i+1-2g} \sigma_{i-2g}^2 \sigma_{i+1-2g}^{-1} \cdots \sigma_{j-2g}^{-1}$, for $i \geq 2g$;

- $A_{2i,j} = \sigma_{j-2g} \cdots \sigma_1 a_{g-i+1} \sigma_1^{-1} \cdots \sigma_{j-2g}^{-1}$, for $1 \leq i \leq g$;
- $A_{2i-1,j} = \sigma_{j-2g} \cdots \sigma_1 b_{g-i+1} \sigma_1^{-1} \cdots \sigma_{j-2g}^{-1}$, for $1 \leq i \leq g$.

Relations (PR1), ..., (PR4) correspond to the classical relations for the pure braid group P_n [Bir]. New relations arise when we consider two generators $A_{2i,j}$, $A_{2i-1,k}$, for $1 \leq i \leq g$ and $j \neq k$. They correspond to loops based at two different points which go around the same handle.

5.2 Mapping class groups, bounding pair braids and pure braids

The *mapping class group* of a surface $\Sigma_{g,p}$, denoted by $\mathcal{M}_{g,p}$, is the group of isotopy classes of orientation-preserving self-homeomorphisms which fix the boundary components pointwise. If the surface has empty boundary then we shall just write $\mathcal{M}_g$. Note that we will denote the composition in the mapping class groups from left to right[1].

Let $\mathcal{P} = \{x_1, \ldots, x_n\}$ be a set of n distinct points in the interior of the surface $\Sigma_{g,p}$. The *punctured mapping class group* of Σ_g relative to $\mathcal{P}$ is defined to be the group of isotopy classes of orientation-preserving self-homeomorphisms which fix the boundary components pointwise, and which fix $\mathcal{P}$ setwise. This group, denoted by $\mathcal{M}_{g,p}^{(n)}$, does not depend on the choice of $\mathcal{P}$, but just on its cardinal. We define the *pure punctured mapping class group*, denoted by $\mathcal{PM}_{g,p}^{(n)}$, to be the subgroup of (isotopy classes of) homeomorphisms which fix the set $\mathcal{P}$ pointwise.

We note T_C a Dehn twist along a simple closed curve C. Let C and D be two simple closed curves bounding an annulus containing the single puncture x_j. We shall say that the multitwist $T_C T_D^{-1}$ is a *j-bounding pair braid*.

Surface braid groups are related to mapping class groups as follows:

THEOREM 17 **(Birman [Bir])** *Let $g \geq 1$ and $p \geq 0$. Let $\psi : \mathcal{M}_{g,p}^{(n)} \longrightarrow \mathcal{M}_{g,p}$ be the homomorphism induced by the map which forgets the set $\mathcal{P}$. If $\Sigma_{g,p}$ is different from the torus then Ker(ψ) is isomorphic to $B_n(\Sigma_{g,p})$.*

REMARK 18 In particular, if Σ is an orientable surface (possibly with boundary) of positive genus and different from the torus, the surface pure braid group $P_n(\Sigma)$ may be identified with the subgroup of $\mathcal{M}_{g,p}^{(n)}$ generated by bounding pair braids (see for instance [Bir], where bounding pair braids are called spin-maps).

[1]We do this in order to have the same group-composition in braid groups and mapping class groups.

5.3 Torelli groups

We recall that the Torelli group $\mathcal{T}_{g,1}$ is the subgroup of the mapping class group $\mathcal{M}_{g,1}$ which acts trivially on the first homology group of the surface $\Sigma_{g,1}$.

Before stating the main theorem of this section, we recall the following exact sequence:

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \mathcal{M}_{g,n+p} \xrightarrow{q} \mathcal{PM}^{(n)}_{g,p} \longrightarrow 1\,, \tag{16}$$

where $\mathbb{Z}^n$ is central and generated by Dehn twists along the first n boundary components of $\Sigma_{g,n+p}$. Geometrically, the projection q may be obtained by glueing one-punctured discs $\mathbb{D}_1, \dots, \mathbb{D}_n$, say, onto the first n boundary components. Note that sequence 16 does not split, since the first homology group of $\mathcal{M}_{g,n+p}$ is trivial when g is greater than 2. Nevertheless, as explained in Lemma 19, when the surface has boundary, we have a section over the corresponding pure braid group.

Lemma 19 *Let $\Sigma_{g,1}$ be a surface of genus greater than or equal to one with one boundary component. Then the group $P_n(\Sigma_{g,1})$ embeds in $\mathcal{T}_{g+n,1}$.*

Proof. Applying Theorem 17 and Remark 18, we identify $P_n(\Sigma_{g,1})$ with the subgroup of $\mathcal{PM}^{(n)}_{g,1}$ generated by bounding pair braids. Let us first embed $P_n(\Sigma_{g,1})$ in $\mathcal{M}_{g,n+1}$. To achieve this, we construct a section s on $P_n(\Sigma_{g,1})$ of the sequence (16). For each generator $A_{i,j}$ of $P_n(\Sigma_{g,1})$, we define $s(A_{i,j})$ as follows. Consider two simple closed curves a and a' lying in $\Sigma_{g,1}$ such that $A_{i,j}$ is equal to the boundary pair braids $T_aT_{a'}^{-1}$. These two curves may be chosen so as to avoid the discs $\mathbb{D}_1, \dots, \mathbb{D}_n$, and thus may be seen as lying in $\Sigma_{g,n+1}$. If d_j is a simple closed curve parallel to the jth-boundary component, we set $s(A_{i,j}) = T_aT_{a'}^{-1}T_{d_j}$, which we denote by $A'_{i,j}$. Since the Dehn twists $T_{d_1}, \dots, T_{d_n}$ belong to the kernel of q, one has $q \circ s = \mathrm{Id}$, and hence s is injective. We claim that s is a homomorphism. To prove this, we have to show that relations (PR1-4) and (ER1-2) are satisfied in $\mathcal{M}_{g,n+1}$ via s.

The four first relations may be written in the form $hA_{r,s}h^{-1} = A_{r,s}$, where h is a word in the $A_{i,j}$'s. These relations are compatible with s, since for all simple closed curves a in $\Sigma_{g,n+1}$, and all h in $\mathcal{M}_{g,n+1}$, one has:

$$T_{h(a)} = h^{-1}T_ah. \tag{17}$$

For example, relation (PR1) is compatible with s because the curves occurring in $A'_{i,j}$ are disjoint from those occuring in $A'_{r,s}$. For (PR2), the bounding pair braid $A_{j,s}$ (resp. $A_{i,j}$, $A_{i,s}$) is equal to $T_{d_j}T_{c_{j,s}}^{-1}$ (resp. $T_aT_{a'}^{-1}$, $T_bT_{b'}^{-1}$ for $(a,a') \in \{(a_i, a_{i,j}), (b_i, b_{i,j}), (d_i, c_{i,j})\}$ and $(b,b') \in \{(a_i, a_{i,s}), (b_i, b_{i,s}), (d_i, c_{i,s})\}$, where curves are those described by Figure 2). Thus we have:

$$\begin{aligned} A'^{-1}_{i,j}A'_{j,s}A'_{i,j} &= [T_a^{-1}T_{a'}T_{d_j}^{-1}]T_{d_j}T_{c_{j,s}}^{-1}T_{d_s}[T_{d_j}T_{a'}^{-1}T_a] \\ &= [T_a^{-1}T_{a'}]T_{c_{j,s}}^{-1}[T_{a'}^{-1}T_a]T_{d_j}T_{d_s} \quad \text{since the } T_{d_k}\text{'s are central} \\ &= T^{-1}_{T_aT_{a'}^{-1}(c_{j,s})}T_{d_j}T_{d_s} \quad \text{by (17),} \end{aligned}$$

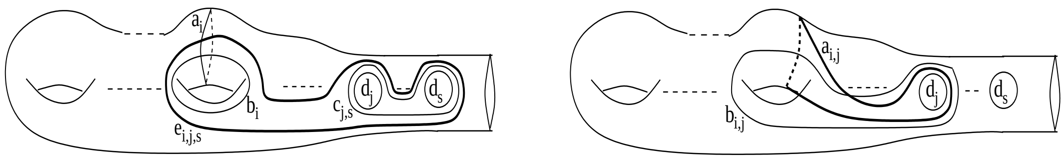

Figure 2: curves for relations (PR2) and (ER2)

and similarly, $A'_{i,s}A'_{j,s}A'^{-1}_{i,s} = T^{-1}_{T_b^{-1}T_{b'}(c_{j,s})}T_{d_j}T_{d_s}$. Now, it is easy to see that

$$T_aT_{a'}^{-1}(c_{j,s}) = T_{a'}^{-1}(c_{j,s}) = T_b^{-1}T_{b'}(c_{j,s}),$$

which yields the required relation. The compatibility of relations (PR3-4) with s may be proved in the same way; we leave this as an exercise for the reader.

Relation (ER1) is a consequence of the lantern relation and relation (17). Indeed, if we consider the seven curves b_i, d_j, d_s, $e_{i,j,s}$, $b_{i,s}$, $b_{j,s}$ and $c_{j,s}$ shown in Figure 2 (where $r = 2i-1$), the lantern relation may be written as:

$$T_{e_{i,j,s}}T_{b_i}T_{d_j}T_{d_s} = T_{b_{i,s}}T_{b_{i,j}}T_{c_{j,s}},$$

which implies that

$$A'_{r,s} = T_{b_i}T^{-1}_{b_{i,s}}T_{d_s} = T_{b_{i,j}}T^{-1}_{e_{i,j,s}}T_{c_{j,s}}T^{-1}_{d_j} = T_{b_{i,j}}T^{-1}_{e_{i,j,s}}T_{d_s}A'^{-1}_{j,s}.$$

Since $A'_{r+1,j} = T_{a_i}T^{-1}_{a_{i,j}}T_{d_j}$, we obtain

$$\begin{aligned}
A'^{-1}_{r+1,j}A'_{r,s}A'_{r+1,j} &= \left[T^{-1}_{d_j}T_{a_{i,j}}T^{-1}_{a_i}T_{b_{i,j}}T^{-1}_{e_{i,j,s}}T_{d_s}T_{a_i}T^{-1}_{a_{i,j}}T_{d_j}\right]\left[A'^{-1}_{r+1,j}A'^{-1}_{j,s}A'_{r+1,j}\right] \\
&= \left[T_{a_{i,j}}T^{-1}_{a_i}T_{b_{i,j}}T^{-1}_{e_{i,j,s}}T_{a_i}T^{-1}_{a_{i,j}}\right]T_{d_s}\left[A'_{r+1,s}A'^{-1}_{j,s}A'^{-1}_{r+1,s}\right] \quad \text{by (PR2)} \\
&= T_{T_{a_i}T^{-1}_{a_{i,j}}(b_{i,j})}T^{-1}_{T_{a_i}T^{-1}_{a_{i,j}}(e_{i,j,s})}T_{d_s}\left[A'_{r+1,s}A'^{-1}_{j,s}A'^{-1}_{r+1,s}\right] \quad \text{by (17).}
\end{aligned}$$

But $T_{a_i}T^{-1}_{a_{i,j}}(b_{i,j}) = b_i$ and $T_{a_i}T^{-1}_{a_{i,j}}(e_{i,j,s}) = b_{i,s}$, so

$$\begin{aligned}
A'^{-1}_{r+1,j}A'_{r,s}A'_{r+1,j} &= T_{b_i}T^{-1}_{b_{i,s}}T_{d_s}\left[A'_{r+1,s}A'^{-1}_{j,s}A'^{-1}_{r+1,s}\right] \\
&= A'_{r,s}A'_{r+1,s}A'^{-1}_{j,s}A'^{-1}_{r+1,s},
\end{aligned}$$

which is relation (ER1). Relation (ER2) is also a consequence of a lantern: again, we leave the details to the reader.

Hence $s : P_n(\Sigma_{g,1}) \longrightarrow \mathcal{M}_{g,n+1}$ is an embedding. Glueing a one-holed torus onto each of the first n boundary components of $\Sigma_{g,n+1}$, we obtain a homomorphism

$\varphi : \mathcal{M}_{g,n+1} \longrightarrow \mathcal{M}_{g+n,1}$ which is injective (see [PR2]). Clearly, the image under φ of each $s(A_{i,j})$ acts trivially on the homology group $H_1(\Sigma_{g+n,1};\mathbb{Z})$. Thus $\varphi \circ s(P_n(\Sigma_{g,1}))$ lies in the Torelli group of $\Sigma_{g+n,1}$. ∎

REMARK 20 This embedding of $P_n(\Sigma_{g,1})$ in $\mathcal{T}_{g+n,1}$ does not hold for surfaces with empty boundary. Indeed, the group $P_n(\Sigma_g)$ has an extra relation (TR) (see Theorem 5.2 of [B]) which is not satisfied by the section s; if L (resp. R) is the left-hand side (resp. right-hand side) of this relation, one can check using lantern relations that we have $s(L) = s(R)d_k^{2(g-1)}$ in $\mathcal{M}_{g,n}$ (k is the same index as in the relation (TR) in Theorem 5.2 of [B]). Nevertheless, it would be interesting to know whether the sequence (16) splits over the pure braid group $P_n(\Sigma_g)$.

THEOREM 21 *Let Σ be the torus, or a surface of positive genus with non-empty boundary. Then the group $P_n(\Sigma)$ is residually torsion-free nilpotent.*

Proof. Let $\Sigma_{g,1}$ be a surface of genus greater than or equal to one, and with one boundary component. First, we remark that Lemma 19 and the residually torsion-free nilpotence of Torelli groups (see for instance section 14 of [H]) imply that $P_n(\Sigma_{g,1})$ is residually torsion-free nilpotent. Now let $\Sigma_{g,p}$ be a surface with $p > 1$ boundary components. The group $P_n(\Sigma_{g,p})$ may be realised as the subgroup of $P_{n+p-1}(\Sigma_{g,1})$ formed by the braids whose first $p-1$ strands are vertical. Therefore $P_n(\Sigma_{g,p})$ is residually torsion-free nilpotent.

The remaining case is that of the pure braid group on n strands of $\mathbb{T}^2$. From Lemma 22, one deduces easily that $D_i(P_n(\mathbb{T}^2)) = D_i(P_{n-1}(\Sigma_{1,1}))$ for $i > 1$, and thus the group $P_n(\mathbb{T}^2)$ is residually torsion-free nilpotent. ∎

LEMMA 22 *The group $P_n(\mathbb{T}^2)$ is isomorphic to $P_{n-1}(\Sigma_{1,1}) \times \mathbb{Z}^2$.*

Proof. Consider the pure braid group exact sequence for an orientable surface Σ:

$$1 \longrightarrow P_{n-1}(\Sigma \setminus \{x_1\}) \longrightarrow P_n(\Sigma) \xrightarrow{\theta} \pi_1(\Sigma) \longrightarrow 1,$$

where geometrically, θ is the map that forgets the paths pointed at $x_2, \ldots, x_n$. Since $ZP_n(\Sigma_{1,1})$ is trivial [PR1], we deduce that the restriction of θ to $ZP_n(\mathbb{T}^2)$ is injective. Since $ZP_n(\mathbb{T}^2) = \mathbb{Z}^2$ [PR1], the restriction of θ to $ZP_n(\mathbb{T}^2)$ is in fact an isomorphism, and we conclude that $P_n(\mathbb{T}^2)$ is isomorphic to the direct product $P_{n-1}(\Sigma_{1,1}) \times \mathbb{Z}^2$. ∎

REMARK 23 In the case of the sphere, the group $P_n(\mathbb{S}^2)$ is isomorphic to $\mathbb{Z}_2 \times P_{n-2}(\Sigma_{0,3})$ (see [GG1]). Therefore, for $i > 1$, $\Gamma_i(P_n(S^2))$ and $\Gamma_i(P_{n-2}(\Sigma_{0,3}))$ are isomorphic. Since $P_{n-2}(\Sigma_{0,3})$ is a subgroup of P_n (which may be realised geometrically as the subgroup of braids whose last strand is vertical), from [FR] it follows that $P_n(\mathbb{S}^2)$ is residually nilpotent, but it is not residually torsion-free nilpotent since $P_n(\mathbb{S}^2)$ has torsion elements.

Remark 24 Pure braid groups of surfaces of genus $g \geq 2$ with empty boundary are bi-orderable ([Go]). To the best of our knowledge, it is not known whether they are residually torsion-free nilpotent.

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